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THEORETICAL ANALYSIS OF THE ONSET OF LIQUID ENTRAINMENT FOR ORIFICES OF FINITE DIAMETER

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Abstract—The onset of liquid entrainment during discharge from large reservoirs containing a stratified mixture of two immiscible fluids through a side orifice of finite diameter is considered theoretically. A previously reported analysis by Craya, in which the orifice was simulated by a point sink, has been extended to account for the finite size of the orifice. The model resulting from the present analysis is expressed in terms of two simple equations suitable for hand calculations. The ratio of the critical height to the orifice diameter is found to be dependent only on Froude number (Fr). The present model approaches the correct physical limits at low Fr and it converges to Craya's predictions at high Fr. Some experimental verification of the present theoretical trends is also provided.

Key Words: theoretical analysis, onset of liquid entrainment, side orifices, finite diameter

1. INTRODUCTION

Theoretical correlation of the onset of liquid entrainment during discharge from side orifices was first developed by Craya (1949). In his simplified analysis, Craya neglected the effects of viscosity and surface tension, assumed potential flow throughout the field and treated the orifice as a point sink. With these idealizations, the following criteria were obtained:

$$\frac{h}{d} = 0.625\,\mathrm{Fr}^{0.4} \qquad [1]$$

and

$$\frac{s}{d} = 0.8\frac{h}{d}, \qquad [2]$$

where d is the orifice diameter, h is the critical height at the onset of the phenomenon, s is the corresponding distance between the tip of the deflected interface and the orifice centreline (see figure 1), and Fr is the Froude number given by

$$\mathrm{Fr} = \frac{V_d}{\sqrt{gd\dfrac{\Delta\rho}{\rho}}} \qquad [3a]$$

with

$$V_d = \frac{W}{\left(\dfrac{\pi}{4}\rho d^2\right)}, \qquad [3b]$$

where g is the gravitational acceleration, ρ is the density of the lighter fluid, $\Delta\rho$ is the density difference between the two fluids, W is the mass flow rate through the orifice and V_d is the mean discharge velocity.

At the lower limit Fr = 0, [1] and [2] predict $h = s = 0$. Evidently, these are not the appropriate limits since the physics of the problem suggest $h = s = d/2$ at Fr = 0. Therefore, the accuracy of [1] and [2] is doubtful at low values of Fr and this behaviour is attributed to the point-sink assumption. On the other hand, for conditions of large discharge rates, Craya's model has been validated by the results of recent experimental investigations (e.g. Crowley & Rothe 1981; Smoglie & Reimann 1986; Schrock *et al.* 1986; Smoglie *et al.* 1987; Micaelli & Memponteil 1989).

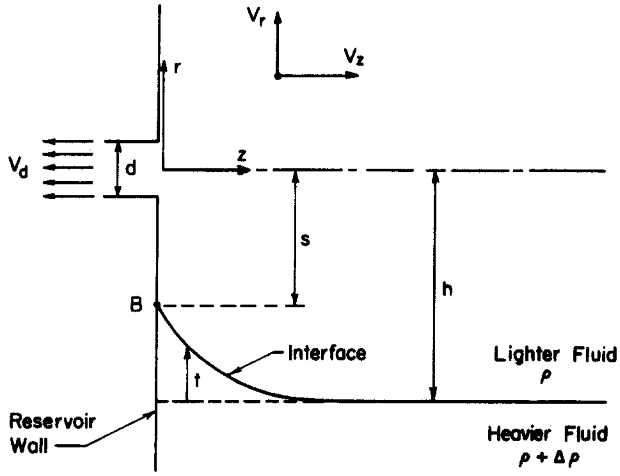

Figure 1. Geometry and the coordinate system.

Another interesting facet relates to the formulation introduced by Smoglie & Reimann (1986) and later adopted by Micaelli & Memponteil (1989), whereby [3a] and [3b] were substituted into [1] to give

$$h\left[\frac{g\rho\,\Delta\rho}{W^2}\right]^{0.2} = \mathrm{K}, \qquad [4]$$

where K is a constant that has the theoretical value of 0.688. According to [4], the critical height h is independent of d for the same g, ρ, $\Delta\rho$ and W. That may be the case when the interface is far from the orifice; however, the general applicability of [4] for all values of h/d is not obvious to the present authors.

The points of concern raised above provided the motivation for this study in which the main objective is to determine the boundary beyond which [1] and [2] can provide good accuracy. In doing so, a more complete model has been developed following Craya's (1949) approach while eliminating the point-sink assumption. The present analysis applies to any two immiscible fluids with the term "liquid entrainment" referring to the appearance of the heavier fluid in the predominantly lighter-fluid flow through the orifice.

2. ANALYSIS

In the present flow situation, the lighter fluid is in motion while the heavier fluid is at rest. The effects of viscosity and surface tension are assumed to be negligible; thus, the inertia and gravity forces are dominant. Steady, incompressible, potential flow is assumed in the lighter fluid and equilibrium of the interface is controlled by a balance between the inertia and gravity forces. The present analysis follows Craya's (1949) approach, where equilibrium of the interface and the velocity field in the lighter fluid are determined first and then equality of the velocity and its gradient at linking point B (see figure 1) are later imposed as conditions for the onset of liquid entrainment.

2.1. Equilibrium of the interface

Applying the Bernoulli equation on a streamline coincident with the interface from the side of the lighter (moving) fluid, we get

$$P + \frac{\rho V^2}{2} + \rho g t = \mathrm{C}, \qquad [5]$$

where P is the static pressure, V is the local velocity, t is the local deflection measured from the flat interface (shown in figure 1) and C is an arbitrary constant. Along the same streamline from the side of the heavier (stationary) fluid, the Bernoulli equation gives

$$P + (\rho + \Delta\rho)gt = \mathrm{C}. \qquad [6]$$

Subtracting [6] from [5], we get

$$\frac{V^2}{2} = \frac{\Delta\rho}{\rho} gt. \qquad [7]$$

Linking point B corresponds to the location on the interface where $t = h - s$. Therefore, at this point

$$\frac{V_B^2}{2} = \frac{\Delta\rho}{\rho} g(h - s). \qquad [8]$$

2.2. *Velocity field in the lighter fluid*

The presence of the stationary fluid is ignored in this part of the analysis. Therefore, the flow field of the lighter fluid is treated as a semiinfinite medium extending over $-\infty < r < \infty$ and $0 \leqslant z < \infty$, where r and z are the coordinates shown in figure 1. Flow within the medium is caused by a discharge with a uniform velocity V_d from the orifice situated at $r \leqslant d/2$ and $z = 0$. Symmetry exists around the z-axis; thus, we have a two-dimensional problem with V_r and V_z representing the velocity components in the r and z directions, respectively. Based on the assumptions stated earlier, the flow field within the medium can be derived from potential flow theory.

Applying the continuity equation, we get

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{\partial V_z}{\partial z} = 0. \qquad [9]$$

Introducing a scalar potential function ϕ, such that

$$V_r = \frac{\partial \phi}{\partial r} \quad \text{and} \quad V_z = \frac{\partial \phi}{\partial z}, \qquad [10]$$

we get the well-known Laplace equation

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{\partial^2 \phi}{\partial z^2} = 0. \qquad [11]$$

Equation [11] is subject to the following list of boundary conditions:

$$\text{at } z = 0, \quad \left.\begin{aligned} \frac{\partial \phi}{\partial z} &= -V_d, \quad 0 \leqslant r \leqslant \frac{d}{2} \\ &= 0 \qquad\quad r > \frac{d}{2} \end{aligned}\right\} \qquad [12a]$$

and

$$\text{as } r \to \infty \quad \text{or} \quad z \to \infty, \quad \phi \text{ is finite.} \qquad [12b]$$

The mathematical formulation given by [11] and [12a, b] is identical to that of heat conduction in a semiinfinite medium with heat extraction at a constant rate over a circular area of diameter d in the plane $z = 0$. By analogy to this problem, for which a solution has been developed by Carslaw & Jaeger (1959), the potential function can be expressed in the form

$$\phi = \frac{V_d d}{2}\int_0^\infty e^{-\lambda z} J_0(\lambda r) J_1\left(\lambda\frac{d}{2}\right)\frac{d\lambda}{\lambda}, \qquad [13]$$

where eigenvalue λ can be of any magnitude form zero to infinity, and J_0 and J_1 are Bessel functions of the first kind. The radial velocity at the wall $z = 0$, $V_{r,0}$, can be determined by substituting [13] into [10]:

$$\frac{V_{r,0}}{V_d} = -\int_0^\infty J_1(\lambda) J_1\left(\frac{2r}{d}\lambda\right) d\lambda. \qquad [14]$$

The integral on the right-hand side of [14] cannot be evaluated in closed form. However, using the tables of integrals by Gradshteyn & Ryzhik (1980), an easier formulation can be developed as

$$\frac{V_{r,0}}{V_{\mathrm{d}}} = -F\left(\frac{r}{d}\right), \qquad [15a]$$

where the function F of any independent variable x is given by

$$F(x) = \frac{1}{4\pi x^2}\int_0^1 \left(\frac{1-y}{y}\right)^{0.5}\left(1-\frac{y}{4x^2}\right)^{-1.5} \mathrm{d}y \qquad [15b]$$

and y is an integration parameter. It is interesting to note that for large values of x, the integral in [15b] approaches the value $\pi/2$ and, therefore, $V_{r,0}/V_{\mathrm{d}}$ approaches $[-1/8(r/d)^2]$ which is identical to the velocity profile due to a point sink. Also, a singularity exists at $r = d/2$ (the edge of the orifice) which, fortunately, does not pose any problem in the subsequent analysis.

Using velocity profile [15a], the kinetic energy at linking point B, located at $r = s$, takes the form

$$\frac{V_{\mathrm{B}}^2}{2} = \frac{V_{\mathrm{d}}^2\left[F\left(\frac{s}{d}\right)\right]^2}{2}. \qquad [16]$$

2.3. The critical height

We now have two expressions for $V_{\mathrm{B}}^2/2$, given by [8] and [16], applicable at point B which links the interface with the wall of the reservoir. For given values of V_{d}, g, ρ, $\Delta\rho$ and d, [16] can be represented by a certain curve on a $V_{\mathrm{B}}^2/2$ vs s plot, while [8] produces a series of parallel straight lines depending on the value of h. For large values of h, the straight line and the curve do not intersect, while two points of intersection are possible with small values of h. There is one value of h that produces a single intersection with the straight line [8] forming a tangent to the curve [16]. Following Craya (1949), this value of h is assumed to be the critical height. Therefore, the onset of liquid entrainment is characterized by the following two non-dimensional relations:

$$\frac{h}{d} - \frac{s}{d} = \tfrac{1}{2}\mathrm{Fr}^2\left[F\left(\frac{s}{d}\right)\right]^2 \qquad [17]$$

and

$$\mathrm{Fr}^2\left[F\left(\frac{s}{d}\right)\right]\left[F'\left(\frac{s}{d}\right)\right] = -1, \qquad [18]$$

where F' is the first derivative of F with respect to s, which has the form

$$F'\left(\frac{s}{d}\right) = -\frac{1}{2\pi\left(\frac{s}{d}\right)^3}\int_0^1 \left(\frac{1-y}{y}\right)^{0.5}\left(1-\frac{y}{4\left(\frac{s}{d}\right)^2}\right)^{-1.5}\left(1-\frac{3y}{2y-8\left(\frac{s}{d}\right)^2}\right)\mathrm{d}y. \qquad [19]$$

For any given value of $s/d > 0.5$, the functions $F(s/d)$ and $F'(s/d)$ can be evaluated from [15b] and [19], respectively, Fr can be determined from [18] and h/d from [17]. Therefore, both h/d and s/d are dependent only on Fr, similar to the form of [1] and [2]. For large values of s/d, the function $F(s/d)$ approaches the value $[0.125/(s/d)^2]$ and $F'(s/d)$ approaches $[-0.25/(s/d)^3]$. Substituting these limiting values into [17] and [18], the present model converges to [1] and [2].

3. RESULTS AND DISCUSSION

3.1. Numerical results

The objective of this section is to compare the results of the present model with those of Craya (1949) in order to determine quantitatively the value of Fr beyond which [1] and [2] are not influenced by the point-sink assumption. Figure 2 shows the velocity profile along the wall in the vicinity of the orifice. The profile given by [15a] approaches infinity at the edge of the orifice and it converges to the point-sink profile away from the orifice. From figure 2, it is clear that the two models give practically identical results beyond $r/d = 2$. Therefore, from a theoretical point of view,

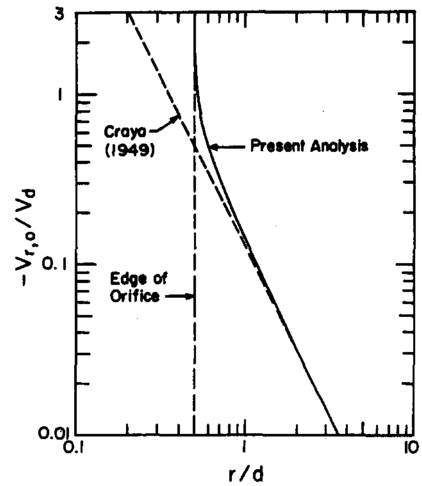

Figure 2. Velocity distribution along the reservoir wall.

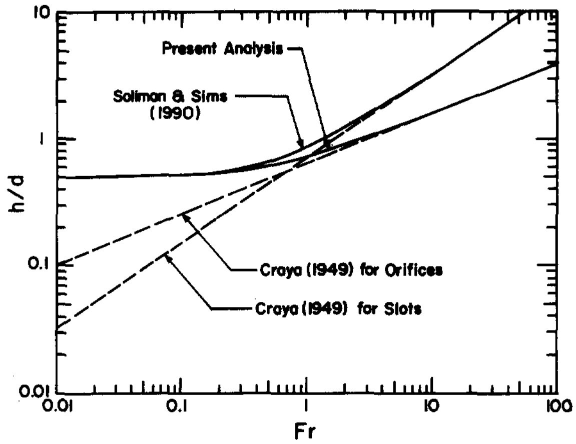

Figure 3. Predictions of h/d for orifices and slots.

the predictions of the present model should be identical to [1] and [2] beyond $s/d = 2$, which corresponds to $h/d = 2.49$ and $\mathrm{Fr} = 30.9$.

The present predictions of h/d at different values of Fr are shown in figure 3. In the same figure, the results from a recent study by Soliman & Sims (1990) for discharge from side slots of width d are shown for comparison. The final correlation from that study characterized the onset of liquid entrainment for side slots of finite width by the following relations:

$$\frac{h}{d} - \frac{s}{d} = \frac{1}{2}\left(\frac{\mathrm{Fr}}{\pi}\right)^2 \left[\ln\left(\frac{\frac{s}{d} - \frac{1}{2}}{\frac{s}{d} + \frac{1}{2}}\right)\right]^2 \qquad [20]$$

and

$$\ln\left(\frac{\frac{s}{d} + \frac{1}{2}}{\frac{s}{d} - \frac{1}{2}}\right) = \frac{\left(\frac{s}{d}\right)^2 - \frac{1}{4}}{\left(\frac{\mathrm{Fr}}{\pi}\right)^2}. \qquad [21]$$

For both orifices and slots, h/d approaches the appropriate limit of 0.5 as Fr approaches 0 and the respective prediction from Craya (1949) is approached at high Fr. The deviation between the present predictions of h/d and those of [1] is 1% at $\mathrm{Fr} = 30.9$, 2.4% at $\mathrm{Fr} = 10$, 16.6% at $\mathrm{Fr} = 1$ and 106% at $\mathrm{Fr} = 0.1$. Experimentally, this deviation is probably large enough to be detected only with $\mathrm{Fr} < 10$.

The behaviour of the ratio s/h at different values of Fr is illustrated in figure 4 for orifices and slots. For both geometries, s/h approaches 1 as Fr approaches 0. At large values of Fr, the ratio s/h converges to 2/3 for slots and 0.8 for orifices, as predicted by Craya (1949).

3.2. *Comparison with experimental data*

In a recent investigation by Armstrong *et al.* (1992, this issue, pp. 217–227), experimental measurements were reported for the critical height at the onset of liquid entrainment during discharge from a large reservoir through a side orifice with $d = 6.35$ mm. The experiment was conducted using stratified air–water mixtures at 310 kPa. A range of discharge flow rates corresponding to $1.7 < \mathrm{Fr} < 31$ was covered in this investigation. Figure 5 shows a comparison between these data and the present analysis in terms of h/h_0 vs Fr, where h_0 is the value obtained from [1] at any given Fr. This figure shows that over the range $8 < \mathrm{Fr} < 31$, the experimental value of h/h_0 is nearly constant with a deviation of about 5.5% from Craya's (1949) model. For $\mathrm{Fr} < 8$, the trend in the experimental data follow the theoretical prediction of increasing h/h_0 with decreasing Fr.

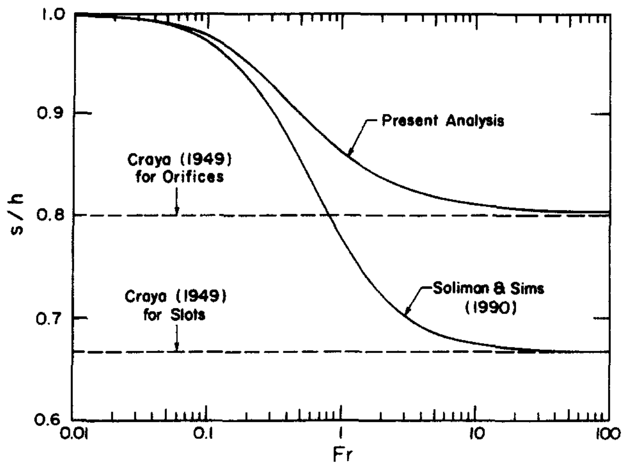

Figure 4. Predictions of s/h for orifices and slots.

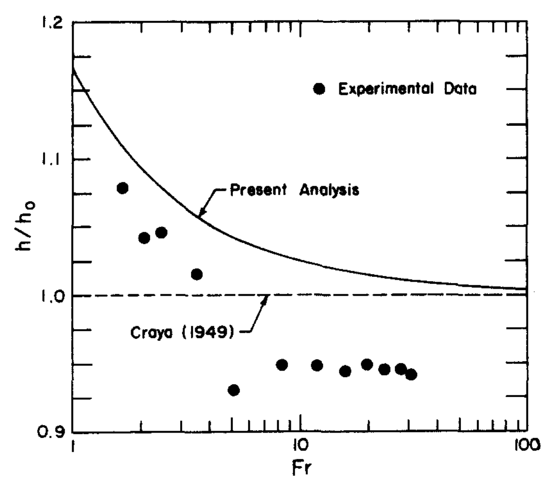

Figure 5. Comparison between the present predictions and experimental data by Armstrong *et al.* (1992).

4. CONCLUDING REMARKS

The present analysis eliminates the point-sink assumption utilized in Craya's (1949) work. A new model is developed for the onset of liquid entrainment from side orifices, given in terms of two simple equations. This model indicates that h/d is uniquely dependent on Fr, irrespective of the value of d. Results of the present analysis are applicable for the whole range of Fr, provided that the assumptions of negligible viscous and surface tension forces are valid. Values of h/d predicted by the present model converge to within 1% of Craya's (1949) predictions for $\mathrm{Fr} \geqslant 30$. Significant deviations between the two models correspond to $\mathrm{Fr} < 10$, as confirmed by recent experimental results.

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